

$$n_i^* = \frac{g_i}{(a + e^{\alpha + \beta \epsilon_i})}$$

F.D.  $a=1$   
 B.E  $a=-1$   
 M.B  $a=0$

$$\ln \omega(i) = n_i \ln \left( \frac{g_i}{n_i} - a \right) - \frac{g_i}{a} \ln \left( 1 - a \cdot \frac{n_i}{g_i} \right)$$

$$\begin{aligned} S/k_B &= \ln S \\ \downarrow \text{Entropy} &= \ln \sum_{\{n_i\}} W(\{n_i\}) \simeq \ln W(n_i^*) = \sum_i \ln \omega(i) (n_i^*) \end{aligned}$$

$$[\ln(x/y) = -\ln(y/x)]$$

$$\begin{aligned} \frac{S}{k_B} &= \sum_i n_i^* \ln \left( \frac{g_i}{n_i^*} - a \right) - \frac{g_i}{a} \ln \left( 1 - a \cdot \frac{n_i^*}{g_i} \right) \\ &= \sum_i \left[ n_i^* \ln \left( g_i + e^{\alpha + \beta \epsilon_i} - a \right) - \frac{g_i}{a} \ln \left( 1 - \frac{a}{a + e^{\alpha + \beta \epsilon_i}} \right) \right] \\ &= \sum_i \left[ n_i^* (\alpha + \beta \epsilon_i) - \frac{g_i}{a} \ln \left( \frac{a + e^{\alpha + \beta \epsilon_i}}{e^{\alpha + \beta \epsilon_i}} \right) \right] \end{aligned}$$

$$\frac{S}{k_B} = \sum_i \left\{ n_i^* (\alpha + \beta \epsilon_i) + \frac{g_i}{\alpha} \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i}) \right\}$$

$$\frac{S}{k_B} = \alpha N + \beta E + \sum_i \frac{g_i}{\alpha} \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i})$$

$$\sum_i \frac{g_i}{\alpha} \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i}) = \frac{S}{k_B} - \alpha N - \beta E$$

$$\alpha = -\frac{\mu}{k_B T}; \beta = \frac{1}{k_B T} \quad (\text{Done earlier})$$

$$\frac{S}{k_B} + \frac{\mu N}{k_B T} - \frac{E}{k_B T} = \sum_i \frac{g_i}{\alpha} \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i})$$

$$TS + \mu N - E = \frac{k_B T}{\alpha} \sum_i g_i \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i})$$

$$\Rightarrow PV = \frac{k_B T}{\alpha} \sum_i g_i \ln(1 + \alpha e^{-\alpha - \beta \epsilon_i})$$

$$\sum_i n_i^* = N$$

$$\alpha \sum_i n_i^* = \sum_i \alpha n_i^* = \alpha N$$

$$\beta \sum_i n_i^* \epsilon_i = \beta E$$

$$\begin{aligned} \mu N &= G \quad (\text{Gibbs Free energy}) \\ &= E + PV - TS \end{aligned}$$

Equation of State for a quantum ideal gas

$$PV = \frac{k_B T}{a} \sum_i g_i \ln(1 + ae^{-d - \beta \epsilon_i})$$

"Small"

$$\simeq \frac{k_B T}{a} \sum_i g_i a e^{-d - \beta \epsilon_i}$$

$$= k_B T \sum_i \underbrace{g_i e^{-d - \beta \epsilon_i}}_{\downarrow n_i^* \text{ (MB)}}$$

In the Maxwell-Boltzmann Case

$$a \rightarrow 0$$

$$\lim_{x \rightarrow 0} \ln(1+x) = x$$

$$= k_B T \sum_i n_i^* = N k_B T$$

$$PV = N k_B T$$

"CLASSICAL"  
ideal gas law  
"MB"

Ideal Bose Gas (non-interacting collection of Bosons)

$$a = -1$$

$$\frac{PV}{k_B T} = - \sum_i g_i \ln(1 - e^{-d - \beta \epsilon_i})$$

$$\frac{PV}{k_B T} = - \sum_i g_i \ln(1 - ze^{-\beta \epsilon_i})$$

Sum is over energy levels

$$e^{-\alpha} \equiv e^{\frac{\mu}{k_B T}} = z$$

↓  
fugacity

Replacing the sum over energy levels with the sum over states and replacing the sum with an integral giving a statistical weight to each state

$$\frac{PV}{k_B T} = - \sum_i g_i \ln(1 - e^{-\alpha - \beta \epsilon_i})$$

[Either high temp or large volume]

$$= - \sum_i g_i \ln(1 - z e^{-\beta \epsilon_i})$$

$g(\epsilon) \cdot d\epsilon \rightarrow$  No. of States bet<sup>n</sup>  $\epsilon$  and  $\epsilon + d\epsilon$

$$\frac{PV}{k_B T} = - \int d\epsilon g(\epsilon) \ln(1 - z e^{-\beta \epsilon})$$

Density of states

$$N = \sum_i \langle n_i \rangle = \sum_i \frac{g_i}{(e^{\alpha + \beta \epsilon_i} - 1)} = \sum_i \frac{g_i}{(z^{-1} e^{\beta \epsilon_i} - 1)}$$

$$N = \int \frac{d\epsilon g(\epsilon)}{(z^{-1} e^{\beta \epsilon} - 1)}$$

How to calculate the density of states?

3D Box  
Particle in

$$E = \epsilon = \frac{(n_x + n_y + n_z)}{8\pi L^2} h^2$$

$$R^2 = \frac{8\pi L^2 E}{h^2} = n_x^2 + n_y^2 + n_z^2$$

No. of States with energy  $\epsilon$

$$\phi(E) = \frac{1}{8} \left( \frac{4}{3} \pi R^3 \right) = \frac{1}{8} \cdot \frac{4}{3} \pi \left( \frac{8\pi L^2 E}{h^2} \right)^{3/2} = \frac{\pi}{6} \left( \frac{8\pi L^2 E}{h^2} \right)^{3/2}$$

only one octant

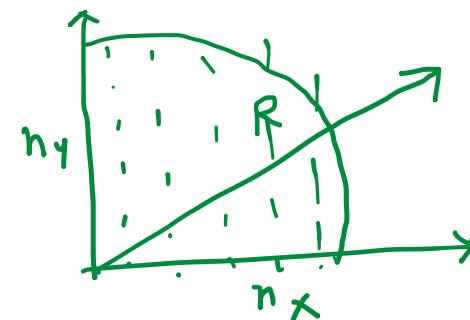
$$\phi(E + \Delta E) =$$

$$\Delta E / E \ll 1 \quad V = L^3$$

$$\omega(E, \Delta E) = \phi(E + \Delta E) - \phi(E)$$

$$\approx \phi(E) + \phi'(E) \Delta E - \phi(E) = \phi'(E) \Delta E$$

$$\begin{aligned} &= \frac{\pi}{6} \left( \frac{8\pi L^2}{h^2} \right)^{3/2} \frac{3}{2} E^{3/2-1} \Delta E \\ &\uparrow = V \\ &= \frac{\pi}{4} \left( \frac{8\pi}{h^2} \right)^{3/2} L^3 E^{1/2} \Delta E \end{aligned}$$



Calculate the no. of lattice points

$$\frac{PV}{k_B T} = - \int \lambda_n (1 - z e^{-\beta E})$$

$$g(E) dE = \left( \frac{2\pi V}{h^3} \right) (2m)^{3/2} \epsilon^{1/2} dE \quad (\text{Please check})$$

$$\langle n_k \rangle_{\text{BE}} = \frac{z e^{-\beta E_k}}{(1 - z e^{-\beta E_k})} \quad \text{BOSONS}$$

$$\langle n_k \rangle_{\text{FD}} = \frac{z e^{-\beta E_k}}{1 + z e^{-\beta E_k}} \quad \text{FERMIANS}$$

What happens if  $T$  is high or density is low ( $g=N/V$ ).  $\Rightarrow$  No. of available energy states  $\gg$  No. of particles  $\rightarrow$  "CLASSICAL"

$$\langle n_k \rangle_{\text{FD/BE}} \approx 0 \quad \Rightarrow \quad z \rightarrow 0 \quad \langle n_k \rangle = z e^{-\beta E_k}$$

(Weakly degenerate case)

$$\sum_k \langle n_k \rangle = z \sum_k e^{-\beta E_k} \underset{z \ll 1}{\approx} z q = N \quad \downarrow \quad z = N/q$$

$$\langle n_k \rangle = z e^{-\beta E_k}$$

$$\langle n_k \rangle = \frac{N}{q} e^{-\beta E_k}$$

$$\frac{\langle n_k \rangle}{N} = \frac{e^{-\beta E_k}}{q} \quad \rightarrow \text{M.B}$$

$$z = e^{\mu/k_B T}$$

$z \rightarrow 0$  (Classical)

$z \gg 0$  (Quantum)

## Equation of State for the Bose gas (ideal Bose)

$$\frac{PV}{k_B T} = - \left( \frac{2\pi V}{h^3} \right) (2m)^{3/2} \int_0^\infty dE \ln(1 - e^{-\beta E}) E^{1/2}$$

### ONE Problem

The state  $E=0$  has been given zero weight  
we should take this state out and then replace the sum with the integral

$$\frac{P}{k_B T} = - \left( \frac{2\pi}{h^3} \right) (2m)^{3/2} \int_0^\infty dE \ln(1 - e^{-\beta E}) E^{1/2} - \frac{\ln(1 - z)}{V}$$

Similarly

$$\frac{N}{V} = \left( \frac{2\pi}{h^3} \right) (2m)^{3/2} \int_0^\infty \frac{E^{1/2}}{(z^{-1} e^{-\beta E} - 1)} + \frac{1}{V} \frac{1}{(z^{-1} - 1)}$$

No. of particles in the ground state  
 $N_0 \approx \frac{z}{1-z}$

$$N = N_e + N_0$$

$$N_e = \left( \frac{2\pi V}{h^3} \right) (2m)^{3/2} \int \dots \quad T \text{ is small} \quad T \rightarrow 0$$

New state of matter  
 As  $T \downarrow$ , the  $E=0$  level gets densely populated.  
**BEC**  
 ~ **Bose-Einstein Condensation**  
 $z \ll 1$  (classical)

$$\frac{1}{V} \left( \frac{z}{1-z} \right) \sim \frac{1}{N} z$$

$\sim$  Vanishing (Small)

$z \gg 0$  ~ approaching 1  
 $z \rightarrow 1$   $\frac{1}{V} \left( \frac{z}{1-z} \right)$  will blow up!!

$$\frac{P}{k_B T} = - \left( \frac{2\pi}{h^3} \right) \frac{(2m)^{3/2}}{\beta^{3/2}} \int_0^\infty dx x^{1/2} \ln(1 - z e^{-x})$$

$$\boxed{\frac{P}{k_B T} = \frac{1}{\lambda^3} g_{3/2}(z)}$$

$$\frac{N_e}{V} = \frac{N - N_0}{V} = \left( \frac{2\pi}{h^3} \right) (2m)^{3/2} \frac{1}{\beta^{3/2}} \int_0^\infty \frac{x^{1/2} dx}{(z^{-1} e^x - 1)}$$

$$\boxed{\frac{N_e}{V} = \frac{1}{\lambda^3} g_{3/2}(z)}$$

$g_\nu(z) \rightarrow$  Bose-Einstein functions

$$g_\nu(z) = \frac{1}{\Gamma(\nu)} \int_0^\infty \frac{x^{\nu-1} dx}{(z^{-1} e^x - 1)} = z + \frac{z^2}{2^\nu} + \frac{z^3}{3^\nu} + \dots$$

$$\lambda = \frac{h}{(2\pi m k_B T)^{1/2}}$$

$$\begin{aligned} \frac{z}{1-z} &= N_0 \Rightarrow \ln(1-z) \\ z &= \frac{N_0}{N_0+1} \quad -\ln(1-z) \frac{1}{V} \\ &= d_n \frac{1}{N_0+1} \\ &= \frac{1}{V} \ln(N_0+1) \\ &\sim \frac{1}{N} \ln N \end{aligned}$$

$\sim 0$  (thermodynamic limit)

$$\left. \begin{aligned} \beta \epsilon &= \frac{\beta^2}{2m k_B T} \\ &= x \\ \beta d\epsilon &= dx \\ d\epsilon &= \frac{1}{\beta} dx \\ \epsilon^{1/2} &= \frac{x^{1/2}}{\beta^{1/2}} \end{aligned} \right\}$$

Upper bound of ( $\nu = 3/2$ )  
 $g_{3/2}(z)$

$$g_{3/2}(1) = g(3/2)$$

$$= 1 + \frac{1}{2^{3/2}} + \frac{1}{3^{3/2}} + \dots$$

$$\{ (3/2) \approx 2.62$$

$$N_e = \frac{V}{\lambda^3} g_{3/2}(z)$$

if fugacity is 1 (quantum)

$$N_e = \frac{V}{\lambda^3} \xi(3/2)$$

$$N_e < \frac{V}{\lambda^3} \xi(3/2)$$

$\downarrow$

$T > 0$

No. of particles in the excited states at  $z=1$  ( $T=0$ )