

$$\hat{H}\psi(x) = E\psi(x)$$

$\psi^*(x, t)\psi(x, t) \rightarrow$ Time independent
 ↳ Stationary State

Region II

$$-\frac{\hbar^2}{2m} \frac{d^2\psi(k)}{dx^2} = E\psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = -\frac{2mE}{\hbar^2} \psi(x)$$

$$\frac{d^2\psi(x)}{dx^2} = -k^2\psi(x)$$

$$\psi(x=0) = \psi(0) = 0 = 0 + B \rightarrow B = 0$$

General Solⁿ

$$\sin kL = 0 \rightarrow$$

$$kL = n\pi \\ n = 0, 1, 2, 3, \dots$$

$$A \sin kx + B \cos kx$$

Also a solⁿ Also a solⁿ

$$A e^{ikx} + B e^{-ikx}$$

$$\psi(x=L) = \psi(L) = 0 = A \sin kL$$

Possible When $A=0 \rightarrow \psi(x)=0$

$$kL = n\pi$$

How to get A?

$$AS_{inkx} = AS \sin\left(\frac{n\pi x}{L}\right) \rightsquigarrow \Psi(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_0^L \Psi^*(x, t) \Psi(x, t) dx = \int_0^L \Psi^*(x) \Psi(x) dx = 1$$

(Normalization condition)

$$\int_0^L \left(A \sin\left(\frac{n\pi x}{L}\right) \right)^* \left(A \sin\left(\frac{n\pi x}{L}\right) \right) dx = 1$$

$$A^2 \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx = 1$$

$$\int \cos x dx = \sin x + C$$

$$\cos 2x = 1 - 2 \sin^2 x$$

$$\Rightarrow \sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

$$1 = \frac{A^2}{2} L \Rightarrow A = \sqrt{\frac{2}{L}}$$

$$1 = \frac{A^2}{2} L - \frac{A^2}{2} \int_0^L \cos\left(\frac{2n\pi x}{L}\right) dx = \frac{L A^2}{2} - 0$$

$$\frac{A^2}{2} \int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right) dx = 1$$

$$\frac{\sin\left(\frac{2n\pi x}{L}\right)}{\left(\frac{2n\pi}{L}\right)} \Big|_0^L = \frac{\sin(2n\pi) - \sin(0)}{(2n\pi/L)} = 0$$

$$k^2 = +\frac{2mE}{\hbar^2} \Rightarrow E_n = \frac{k^2 \hbar^2}{2m} = \frac{n^2 \pi^2 \hbar^2}{2mL^2} \sim \text{Energy} \quad \hbar = h/2\pi$$

$$kL = n\pi$$

$$\Psi_1 = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right)$$

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$x = \frac{L}{2}$

ground state
wavefunction

$$\Psi_2 = \sqrt{\frac{2}{L}} \sin\left(\frac{2\pi x}{L}\right)$$

$$x = \frac{L}{2} \quad \Psi_2 = 0$$

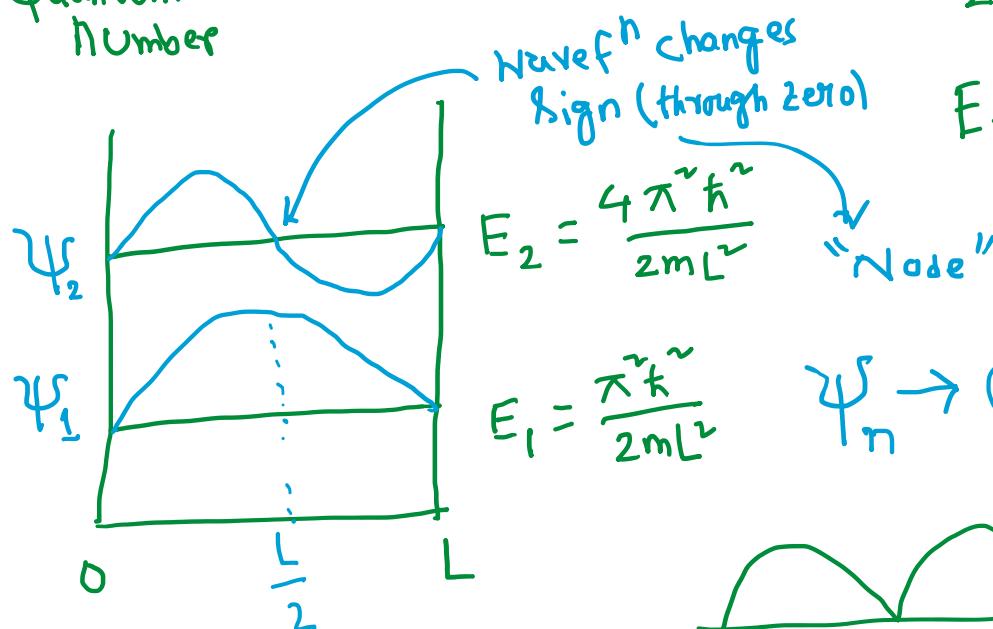
$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2} = \frac{n^2 \hbar^2}{8mL^2}$$

Quantum Number

$$E_1 = \frac{\hbar^2}{8mL^2} = \frac{\pi^2 \hbar^2}{2mL^2}$$

$$E_2 = \frac{4\pi^2 \hbar^2}{2mL^2} = \frac{2\pi^2 \hbar^2}{mL^2}$$

$$E_3 = 9E_1$$



$\langle x \rangle_1 \rightarrow$ Expectation value of x in the ground state (ψ_1)

$$\langle x \rangle_1 = \frac{\int_0^L \psi_1^*(x) x \psi_1(x) dx}{\int_0^L \psi_1^*(x) \psi_1(x) dx}$$

$$\psi_1^*(x) = \psi_1(x)$$

$$\langle x \rangle_1 = \int_0^L x \psi_1(x) dx = 0$$

$$\langle x \rangle_n = 0$$

$$\begin{aligned} \langle x \rangle_1 &= \frac{2}{L} \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L} \int_0^L \frac{x}{2} (1 - \cos\left(\frac{2n\pi x}{L}\right)) dx \\ &= \frac{2}{L} \cdot \frac{1}{2} \frac{x^2}{2} \Big|_0^L - \frac{1}{L} \int_0^L x \cos\left(\frac{2n\pi x}{L}\right) dx \\ &= \frac{L}{2} - 0 \end{aligned}$$

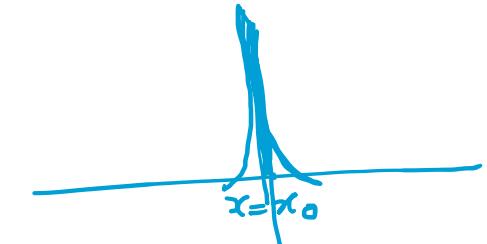
$$\Psi_n(x, t) = \underbrace{\psi_n(x)}_{\text{Stationary State}} e^{-i E_n t / \hbar}$$

Also $\psi_n(x)$ is normalized

$\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$ is not an eigenfunction
of position operator \hat{x}

$$x \sin\left(\frac{n\pi x}{L}\right) \neq \text{a constant } \sin\left(\frac{n\pi x}{L}\right)$$

$$\begin{aligned} &\delta(x - x_0) \\ &\text{Eigen f. of } \hat{x} \\ &\int_{-\infty}^{\infty} x \delta(x - x_0) = x_0 \end{aligned}$$



$$I = \int_0^L x \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$= x \sin\left(\frac{2n\pi x}{L}\right) \Big|_0^L - \int_0^L \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= 0$$

$$\langle x \rangle_n = \frac{L}{2}$$

$$\hat{p}_x = -ik \frac{\partial}{\partial x} = -ik \frac{d}{dx}$$

$$\langle \hat{p}_x \rangle_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(-ik \frac{d}{dx}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= -\left(\frac{2}{L}\right)(ik) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \frac{d}{dx} \left(\sin\left(\frac{n\pi x}{L}\right)\right) dx = \left(\cdots\right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \cancel{\cos\left(\frac{n\pi x}{L}\right)} dx = 0$$

$$\int u dv = uv - \int v du$$

$$\frac{d}{dx} \cos x = -\sin x$$

$$\int \cos x \frac{du}{dx} = \sin x + C$$

$$\int_0^L \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= - \frac{\cos\left(\frac{2n\pi x}{L}\right)}{\frac{2n\pi}{L}} \Big|_0^L$$

$$= -\frac{L}{2n\pi} \left\{ \cos\left(2n\pi\frac{1}{2}\right) - \cos\left(2n\pi\frac{0}{2}\right) \right\}$$

$$= 0$$

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = 0$$

$$\begin{aligned}
 & \langle p_x^n \rangle_n \\
 & p_x^n = \left(-i\hbar \frac{d}{dx} \right) \left(-i\hbar \frac{d}{dx} \right) = \left(\frac{2}{L} \right) \int_0^L \sin\left(\frac{n\pi x}{L}\right) (-\hbar^2) \frac{d^2}{dx^2} \left\{ \sin\left(\frac{n\pi x}{L}\right) \right\} dx \\
 & p_x^n = -\hbar^2 \frac{d^2}{dx^2} \\
 & = -\left(\frac{2}{L} \right) \hbar^2 \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(-\frac{n^2 \pi^2}{L^2} \right) \sin\left(\frac{n\pi x}{L}\right) dx \\
 & = \left(\frac{2}{L} \right) \hbar^2 \frac{n^2 \pi^2}{L^2} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\
 & \int_0^L \psi_n^2(x) dx = 1 \\
 & \langle p_x^n \rangle = \frac{n^2 \pi^2 \hbar^2}{L^2} \\
 & K.E = \frac{\langle p_x^2 \rangle}{2m} = \frac{n^2 \pi^2 \hbar^2}{2m L^2} = E_n
 \end{aligned}$$

$$\langle x^2 \rangle_n = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) x^2 \sin\left(\frac{n\pi x}{L}\right) dx \neq 0$$

$$\hat{p}_x \psi_n(x) = -i\hbar \frac{d}{dx} \left\{ \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right\} = -i\hbar \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L} \right) \cos\left(\frac{n\pi x}{L}\right)$$

$\neq \text{constant } \psi_n(x)$

$\psi_n(x)$ is not an eigenfunction
of \hat{p}_x .

But,

$$\begin{aligned} \hat{p}_x^2 \psi_n(x) &= \hat{p}_x \hat{p}_x \psi_n(x) \\ &= -i\hbar \sqrt{\frac{2}{L}} \left(\frac{n\pi}{L} \right) \left(\frac{n\pi}{L} \right) i(-\sin\left(\frac{n\pi x}{L}\right)) \quad \hat{p}_x^2 \text{ (eigenvalue)} \end{aligned}$$

$$\begin{aligned} &= \frac{n^2 \pi^2 \hbar^2}{L^2} \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \quad \psi_n(x) \\ &\quad = \left(\frac{n^2 \pi^2 \hbar^2}{L^2} \right) \psi_n(x) \end{aligned}$$

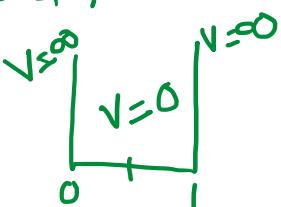
$\hat{p}_x^2 \rightarrow \psi_n(x)$ is an eigenfunction of \hat{p}_x^2 ($\hat{p}_x^2 / 2m = \hat{H}$)

$$E_n = \frac{n^2 \hbar^2 \pi^2}{2mL^2} ; n = 1, 2, 3, \dots$$

$$E_1 = \frac{\hbar^2 \pi^2}{2mL^2}$$

Zero point energy

$$\psi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right)$$



$$\langle \hat{x} \rangle = \frac{L}{2} \quad \langle \hat{p}_x \rangle = \frac{n^2 \hbar^2 \pi^2}{L^2}$$

$$\langle \hat{p}_x \rangle = 0$$

$$\langle \hat{x} \rangle = ?$$

$$\begin{aligned} \langle \hat{p}_x \rangle &= \int_0^L \left(\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right)^* \left(-i\hbar \frac{d}{dx} \right) \left(\sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) \right) dx \\ &= \left(\frac{2}{L} \right) (-i\hbar) \int_0^L \sin\left(\frac{n\pi x}{L}\right) \left(\frac{n\pi}{L} \right) \cos\left(\frac{n\pi x}{L}\right) dx \\ &= \left(\frac{2}{L} \right) (-i\hbar) \int_0^L \sin\theta \cos\theta d(\sin\theta) \\ &= \frac{1}{2} \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx \\ &= \frac{1}{2} \left[\left(\sin\left(n\pi\right) - \sin 0 \right) \right] \end{aligned}$$

$$\langle x^2 \rangle = \frac{2}{L} \int_0^L dx \sin\left(\frac{n\pi x}{L}\right) x^2 \sin\left(\frac{n\pi x}{L}\right) = \frac{2}{L} \int_0^L x^2 \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$2\sin^2 x = 1 - \cos(2x)$$

$$= \frac{2}{L} \frac{1}{2} \int_0^L x^2 (1 - \cos\left(\frac{2n\pi x}{L}\right)) dx$$

$\int u dv = uv - \int v du$
Integration by parts

$$= \frac{1}{L} \int_0^L x^2 dx - \frac{1}{L} \int_0^L x^2 \cos\left(\frac{2n\pi x}{L}\right) dx$$

$$= \frac{1}{L} \frac{L^3}{3} - \frac{1}{L} I$$

$$I = \left[x^2 \sin\left(\frac{2n\pi x}{L}\right) \left(\frac{1}{\frac{2n\pi}{L}} \right) \right]_0^L - \int_0^L \frac{\sin\left(\frac{2n\pi x}{L}\right)}{\left(\frac{2n\pi}{L}\right)} 2x dx$$

$$\langle x^2 \rangle = \frac{L^2}{3} - \frac{1}{L} \cdot \frac{L^3}{2n^2\pi^2} = \frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} = L^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right)$$

$$I = -\frac{2L}{2n\pi} \int_0^L x \sin\left(\frac{2n\pi x}{L}\right) dx$$

$$= -\frac{L}{n\pi} \left[-x \frac{\cos\left(\frac{2n\pi x}{L}\right)}{\frac{2n\pi}{L}} \right]_0^L + \int_0^L \frac{\cos\left(\frac{2n\pi x}{L}\right)}{\left(\frac{2n\pi}{L}\right)} dx$$

$$\rightarrow \left(\frac{L}{2n\pi} \right)^2 \sin\left(\frac{2n\pi x}{L}\right) \Big|_0^L = \frac{L^2}{2n^2\pi^2} \left[L \cos\left(\frac{2n\pi L}{2n\pi}\right) - 0 \right] = \frac{L^3}{2n^2\pi^2}$$

$$\langle x \rangle = \frac{L}{2}$$

$$\langle x^2 \rangle = L^2 \left(\frac{1}{3} - \frac{1}{2n^2\pi^2} \right)$$

$$\langle \hat{p}_x \rangle = 0$$

$$\langle \hat{p}_x^2 \rangle = \frac{n^2 \hbar^2 \pi^2}{L^2}$$

$$\sigma_x = \sqrt{\frac{L^2}{3} - \frac{L^2}{2n^2\pi^2} - \frac{L^2}{4}} = \sqrt{\frac{L^2}{12} - \frac{L^2}{2n^2\pi^2}}$$

$$\sigma_{\hat{p}_x} = \sqrt{\frac{n^2 \hbar^2 \pi^2 / L^2}{L}} = \frac{n \hbar \pi}{L}$$

Real and > 0

(Square of a real number)

$$\sigma_x^2 = \langle (x - \langle x \rangle)^2 \rangle \rightsquigarrow \text{Mean Square deviation} \rightsquigarrow \text{Variance}$$

$$\langle (x^2 - 2x\langle x \rangle + \langle x \rangle^2) \rangle = \langle x^2 \rangle - 2\langle x \rangle \langle x \rangle + \langle x \rangle^2$$

$$= \langle x^2 \rangle - 2\langle x \rangle^2 + \langle x \rangle^2 = \langle x^2 \rangle - \langle x \rangle^2, 0$$

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2}$$

Root mean square

deviation / Standard

deviation (Δx)

\downarrow uncertainty

$$\sigma_{\hat{p}_x} = \sqrt{\langle \hat{p}_x^2 \rangle - \langle \hat{p}_x \rangle^2}$$

$$\sigma_x \sigma_{\hat{p}_x} = \frac{n \hbar \pi}{L} \cdot \sqrt{\frac{L^2}{4n^2\pi^2} \left(\frac{n^2\pi^2}{3} - 2 \right)}$$

$$= \frac{n \pi \hbar}{L} \cdot \frac{\hbar}{2} \sqrt{\left(\frac{n^2\pi^2}{3} - 2 \right)}$$

$$\sigma_x \sigma_{\hat{p}_x} = \hbar / 2 \sqrt{\left(\frac{n^2\pi^2}{3} - 2 \right)}$$

$$\Delta x \langle \hat{p}_x \rangle, \hbar / 2$$

$$\sigma_x \sigma_{\hat{p}_x}, \hbar / 2$$

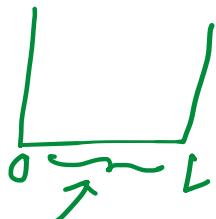
$$\sqrt{\left(\frac{n^2\pi^2}{3} - 2 \right)} = 1.13572 > 1$$

$$\int_0^L dx \psi_n^*(x) \psi_m(x) = \frac{2}{L} \int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx$$

Not mass
but quantum
number here

$$= \frac{2}{L} \frac{1}{2} \int_0^L \left(\cos\left(\frac{(n-m)\pi x}{L}\right) - \cos\left(\frac{(n+m)\pi x}{L}\right) \right) dx$$

Completeness



$$f(x) = c_1 \psi_1 + c_2 \psi_2 + \dots$$

Any $f(x)$ for $0 < x < L$ $\{c_i\} = 0$

If $n=m$

$$= 1 \quad (\text{normalization})$$

ψ_n 's form an orthonormal set

Each of them normalized and orthogonal to each other

$n \neq m$

$$2 \sin \theta \sin \phi = \left\{ \begin{array}{l} \cos(\theta - \phi) \\ - \cos(\theta + \phi) \end{array} \right\}$$

$$= \frac{1}{L} \left[\frac{\sin\left(\frac{(n-m)\pi x}{L}\right)}{(n-m)\pi/L} \Big|_0^L - \frac{\sin\left(\frac{(n+m)\pi x}{L}\right)}{(n+m)\pi/L} \Big|_0^L \right]$$

$$\int_0^L dx \psi_n^*(x) \psi_m(x) = \delta_{nm}$$

$\delta_{nm} = 1 \quad \text{if } n=m$
 $= 0 \quad \text{if } n \neq m$

Kronecker delta

Eigenfns of \hat{H} form an orthonormal set

Because \hat{H} is Hermitian

$$\sigma_x \sigma_{p_x} \geq \hbar/2$$

↓ position ↓ conjugate momentum

$$\hat{x} \quad \hat{p}_x = -i\hbar \frac{\partial}{\partial x} = -i\hbar \frac{d}{dx}$$

$$\hat{x} f(x) = x f(x)$$

$$\hat{p}_x (\hat{x} f(x)) = \left(-i\hbar \frac{d}{dx}\right) x f(x) = (-i\hbar) \left\{ f(x) + x \left(\frac{df(x)}{dx} \right) \right\}$$

$$= (-i\hbar) \left\{ f(x) + x f'(x) \right\}$$

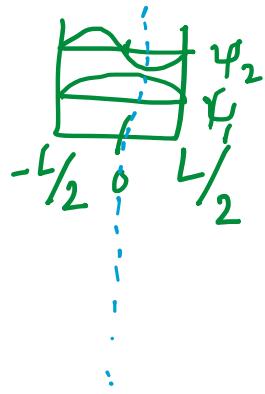
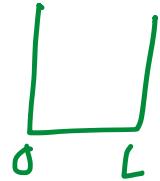
$$\hat{x} \hat{p}_x f(x) = x (-i\hbar) f'(x)$$

$$[\hat{x} \hat{p}_x - \hat{p}_x \hat{x}] f(x) = \cancel{x (-i\hbar) f'(x)} + i\hbar f(x) + \cancel{(i\hbar) x f'(x)}$$

$$[\hat{x}, \hat{p}_x] f(x) = i\hbar f(x)$$

Commutator $\rightarrow [\hat{x}, \hat{p}_x] = i\hbar \neq 0 \rightarrow \hat{x}, \hat{p}_x$ they do not commute \rightarrow they do not have any common eigen f.n. \rightarrow Uncertainty

$[\hat{A}, \hat{B}] = 0$
then they will have a common set of eigen functions



$$V(x) = 0 \quad -\frac{L}{2} < x < \frac{L}{2}$$

$$= \infty \quad \text{otherwise}$$

$$V(-x) = V(x) \quad \text{Symmetric or even f^n of x}$$

$\psi_1 \rightarrow$ Also even fⁿ of x

$$\psi_1(-x) = \psi_1(x)$$

$$\psi_2(-x) = -\psi_2(x) \quad \text{odd f^n of x}$$

Antisymmetric

$$\hat{H}(x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x)$$

$$\hat{H}(-x) = -\frac{\hbar^2}{2m} \frac{d^2}{d(-x)^2} + V(-x)$$

$$\hat{H}(-x) = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) = \hat{H}(x)$$

$$\hat{H}\psi_n(x) = E_n \psi_n(x) \rightarrow$$

this should also be valid for -x

$$\hat{H}\psi_n(-x) = E_n \psi_n(-x) \quad \left| \begin{array}{l} \text{How it is} \\ \text{coming} \end{array} \right.$$

$$\psi_n(-x) = c \psi_n(x) \quad \left| \begin{array}{l} \text{and what are the} \\ \text{possible values of } c \end{array} \right.$$

$$\hat{H}(-x)\psi_n(x) = E_n \psi_n(x)$$

$$\hat{H}(-x)\psi_n(-x) = E_n \psi_n(-x) \rightarrow$$

Also an eigenfⁿ of \hat{H}

$$\psi_n(-x) = c \psi_n(x)$$

$$c = 1 \text{ or } c = \pm 1$$

$$\psi_n(x) = \psi_n(-(-x)) = c \psi_n(-x) \rightarrow$$

$$= c^2 \psi_n(x)$$

$$\frac{1}{\sqrt{2}} (\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}) = \phi$$

Not a stationary state

$$\phi^* \phi \rightarrow \text{time dependent}$$

$$\begin{aligned} & \frac{1}{\sqrt{2}} (\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar})^* \rightarrow e^{iE_1 t/\hbar} \\ & \frac{1}{\sqrt{2}} (\psi_1^* e^{-iE_1 t/\hbar} + \psi_2^* e^{-iE_2 t/\hbar}) \end{aligned}$$

$$= \frac{1}{2} \left[\psi_1^* \psi_1 + \psi_2^* \psi_2 + \psi_2^* \psi_1 e^{+\frac{i}{\hbar} (E_2 - E_1) t} + \psi_1^* \psi_2 e^{-\frac{i}{\hbar} (E_2 - E_1) t} \right]$$

f" of time